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A note on the energetics of a double-diffusive system

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Adjacent oceanic water masses with the same density but different concentrations of heat and salt generate interleaving accompanied by double-diffusive processes. Laboratory experiments with salt and sugar concentrations are used to study the interleaving process. Most double-diffusive studies have treated vertical configurations in which one of the two components contains a destabilizing feature, salt above fresh water for salt fingers or warm underlying cold for the diffusive case. However, when the fluid lacks any gravitationally unstable feature, i.e. no gravitational potential energy is available in either component, the question arises as to what the source of energy is to drive the system. Such a case is discussed here and it is shown that the ultimate source of the energy is the chemical potential associated with the different property distributions. Diffusion creates a destabilizing property distribution and then enables the resulting potential energy to be released.

1. Introduction

In a recent article Thompson & Veronis (2005) reported the results of laboratory experiments on double diffusion in which the initial state involved distributions of salt (T) and sugar (S), both of which were gravitationally stable. The experimental layer had a uniform layer of salt on one side and a uniform layer of sugar of the same density adjacent to it, the two separated by a vertical barrier at the midpoint. The layer sat on a reservoir of greater density in both sugar and salt. The experiment started when the barrier was withdrawn and motion ensued because salt, which has the larger diffusivity coefficient, diffused up from the reservoir thereby making the bottom of the sugar layer denser. Consequently, this denser layer slid under the salt layer and generated an overturning circulation in which the interface dividing the two layers gradually tilted and flattened until the configuration became one of two layers with the lighter (mostly salt) layer on top. Figure 1 is a photo of one such experiment some time after the barrier was removed.

A similar, but somewhat weaker, motion can be observed when the underlying reservoir is replaced by a solid boundary, as Ruddick & Turner (1979) observed. In both cases the question that arises is how the kinetic energy of the system is generated given that the gravitational potential energy is initially at a minimum, so that a vertical exchange of parcels would not release potential energy. The purpose of this note is to try to answer that question by evaluating the energy-releasing term using the diffusion equations for salt and sugar in the gravitational term. The model used has no bottom reservoir since it is simpler but still contains the essential features. The analysis is for a two-dimensional system with x horizontal and z vertical.

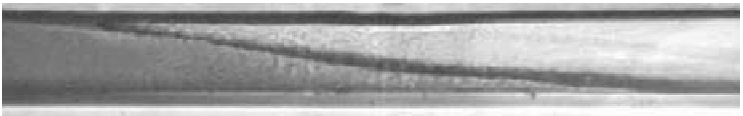

FIGURE 1. The dark fluid is the sugar solution that has become denser because of downward diffusion of salt from the overlying salt solution. The slanting interface is nearly linear and the fronts penetrate laterally with nearly constant velocity.

2. Analysis of the energetics

The Boussinesq approximation is assumed to be valid so that the same velocity is used for all of the equations; otherwise, the equations for the conservation of T and S would involve different mass-weighted velocities. In the following development the density, ρ, which appears in the gravitational term, refers to the density associated with the S and T concentrations. The kinetic energy equation integrated vertically from bottom to top ($-h$ to 0) and horizontally from left to right ($-L$ to L) and in time (0 to t) is

$$\int_{-L}^{L}\int_{-h}^{0}\frac{1}{2}\boldsymbol{u}\cdot\boldsymbol{u}\,\mathrm{d}z\,\mathrm{d}x=-\int_{0}^{t}\int_{-L}^{L}\int_{-h}^{0}\frac{g\rho w}{\rho_{00}}\,\mathrm{d}z\,\mathrm{d}x\,\mathrm{d}t-\int_{0}^{t}\int_{-L}^{L}\int_{-h}^{0}\nu\nabla\boldsymbol{u}:\nabla\boldsymbol{u}\,\mathrm{d}z\,\mathrm{d}x\,\mathrm{d}t, \quad (1)$$

where ρ_{00} is the constant density of the mean state, and the boundary conditions are zero velocity along the sides and the bottom and zero shear and zero vertical velocity along the top. (To lowest order the free surface is assumed to be flat.) The boundaries are non-diffusive, i.e. $\partial T/\partial n=0=\partial S/\partial n$. A cursory inspection of the gravitational term in (1) indicates that no potential energy can be released because there is no lighter fluid that can be carried upward. Clearly, a more subtle treatment of that term is required to show that gravity releases potential energy.

In order to evaluate the (gravitational) energy releasing term we shall use a linear approximation to the equation of state

$$\rho=\alpha T+\beta S \quad (2)$$

where α/ρ_{00} and β/ρ_{00} are coefficients of contraction of T and S and use of α and β as constants involves an error of less than 1 % for the range of values of T and S in the experiments. From here on we shall use the symbols T and S to represent the concentration densities, αT and βS, so that $\rho=T+S$.

The equations for the conservation of salt and sugar provide the means for evaluating the gravitational term:

$$\frac{\partial T}{\partial t}+\frac{\partial(uT)}{\partial x}+\frac{\partial(wT)}{\partial z}=\kappa_T\nabla^2 T, \quad (3)$$

$$\frac{\partial S}{\partial t}+\frac{\partial(uS)}{\partial x}+\frac{\partial(wS)}{\partial z}=\kappa_S\nabla^2 S. \quad (4)$$

(With constant α and β these equations apply to αT and βS as well as to T and S.) In the experiments the right half of the tank is initially occupied by the concentration, $T=T_0$, and the left half by $S=S_0$, and the density of each concentration equals ρ_0. So integrating with respect to x and t and taking the side boundaries to be non-diffusive yields

$$\int_{-L}^{L} T\,\mathrm{d}x - LT_0 + \frac{\partial}{\partial z}\int_0^t\int_{-L}^{L} wT\,\mathrm{d}x\,\mathrm{d}t = \kappa_T\frac{\partial^2}{\partial z^2}\int_0^t\int_{-L}^{L} T\,\mathrm{d}x\,\mathrm{d}t, \tag{5}$$

$$\int_{-L}^{L} S\,\mathrm{d}x - LS_0 + \frac{\partial}{\partial z}\int_0^t\int_{-L}^{L} wS\,\mathrm{d}x\,\mathrm{d}t = \kappa_S\frac{\partial^2}{\partial z^2}\int_0^t\int_{-L}^{L} S\,\mathrm{d}x\,\mathrm{d}t. \tag{6}$$

Next, integrating with respect to z from z to 0 yields

$$\int_z^0\int_{-L}^{L} T\,\mathrm{d}x\,\mathrm{d}z + LzT_0 - \int_0^t\int_{-L}^{L} wT\,\mathrm{d}x\,\mathrm{d}t = -\kappa_T\frac{\partial}{\partial z}\int_0^t\int_{-L}^{L} T\,\mathrm{d}x\,\mathrm{d}t, \tag{7}$$

$$\int_z^0\int_{-L}^{L} S\,\mathrm{d}x\,\mathrm{d}z + LzS_0 - \int_0^t\int_{-L}^{L} wS\,\mathrm{d}x\,\mathrm{d}t = -\kappa_S\frac{\partial}{\partial z}\int_0^t\int_{-L}^{L} S\,\mathrm{d}x\,\mathrm{d}t. \tag{8}$$

Now adding (7) and (8), rearranging terms and making use of the linear equation of state, $\rho = T + S$, yields

$$\begin{aligned}\int_0^t\int_{-L}^{L} w\rho\,\mathrm{d}x\,\mathrm{d}t = {}& \int_z^0\int_{-L}^{L} \rho\,\mathrm{d}x\,\mathrm{d}z + 2zL\rho_0 \\ & + \kappa_T\frac{\partial}{\partial z}\int_0^t\int_{-L}^{L} T\,\mathrm{d}x\,\mathrm{d}t + \kappa_S\frac{\partial}{\partial z}\int_0^t\int_{-L}^{L} S\,\mathrm{d}x\,\mathrm{d}t,\end{aligned} \tag{9}$$

where we have used the fact that $T_0 = S_0 = \rho_0$, since each original half-layer contains only one component, each of which has the same density, ρ_0.

And finally, integrating with respect to z from $-h$ to 0 yields the required expression for the energy-releasing term

$$\begin{aligned}\int_{-h}^0\int_0^t\int_{-L}^{L} w\rho\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}z = {}& \int_{-h}^0\int_z^0\int_{-L}^{L} \rho\,\mathrm{d}x\,\mathrm{d}z'\,\mathrm{d}z - h^2L\rho_0 \\ & + \left[\kappa_T\int_0^t\int_{-L}^{L} T\,\mathrm{d}x\,\mathrm{d}t + \kappa_S\int_0^t\int_{-L}^{L} S\,\mathrm{d}x\,\mathrm{d}t\right]_{-h}^{0}.\end{aligned} \tag{10}$$

We have no theoretical solution to this system so we rely on experimental data to evaluate the terms on the right-hand side of (10). Marshall Ward and the author carried out a number of lock exchange experiments during the 2004 GFD program at WHOI. The ones without rotation are similar to those by Thompson & Veronis (2005) but there is no bottom reservoir. A preliminary report is in Ward (2004). Additional experiments were conducted during the summer program of 2005, from which the measurements of S and T reported below have been taken. Throughout each experiment diffusion of both salt and sugar takes place across the sloping interface, and since salt diffuses faster than sugar, there is a decrease in density in the upper (salty) layer and an increase in density in the lower layer.

We obtained vertical profiles of the conductivity of the fluid at the centre location and measured the refractive index at the top and bottom. Therefore, we do not have a profile of the density but only the values at the top and bottom. We obtain a conservative estimate of the redistribution of density by approximating the profile by a straight line that joins the top and bottom measured values, which are $\rho_0-\Delta\rho/2$ and $\rho_0 + \Delta\rho/2$, respectively. (This is an underestimate of the density difference between the upper and lower parts. We made some crude measurements of the densities near the interface and found that both the upper and the lower parts were more uniform vertically than a linear profile indicates. Thus, choosing a linear profile is

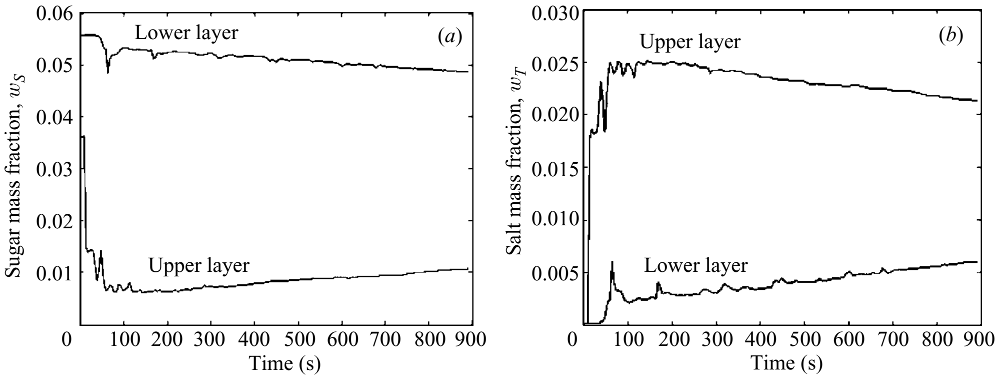

FIGURE 2. (*a*) Mass fraction of S and (*b*) T near the top of the upper layer and the bottom of the lower layer at the middle position in the experiment as a function of time. Note the nearly linear dependence on time after an initial adjustment.

conservative.) Therefore, the profile from $-h$ to 0 is given by

$$\rho_0 - \Delta\rho(1/2 + z/h),$$

and the first integral on the right-hand side of (10) is

$$L[h^2\rho_0 - \Delta\rho h^2/6].$$

The ρ_0 terms on the right-hand side of (10) cancel so the net result of the first two terms is

$$-L[\Delta\rho h^2/6]. \tag{11}$$

The value of $\Delta\rho$ is not known but the important issue is the sign of the term. The two diffusion terms in (10) were evaluated using the data in figure 2 which shows S and T values near the top and bottom boundaries as functions of time up to $t = 900\,\mathrm{s}$, by which time the sloping interface had intersected the ends and had become nearly horizontal. Crudely, the effect of the integration over the horizontal is tantamount to multiplication by $2L$ (which is an overestimate), and since the values at the top and bottom depend essentially linearly on time (apart from a very noisy initial disturbance which contributes little to the integrals), the diffusion terms can be evaluated easily. They contribute a positive value to the right-hand side of (10) and, therefore, counteract the effect of (11). The diffusivities are very small (0.5 and $1.5 \times 10^{-5}\,\mathrm{cm^2\,s^{-1}}$ for sugar and salt, respectively) and the net result is that the two diffusion terms turn out to be about 2% of the value given by (11). This is an overestimate of the effect of diffusion at the top and bottom boundaries during the times shown so the overall result, including (11), is a conservative estimate of the energy-releasing term. (Of course, over a very much longer period the (usual) effects of diffusion will serve to smooth out all gradients.)

Thus, substituting the negative term (11) for $w\rho$ on the right-hand side of (1) yields a positive total energy-releasing term, which serves to balance part of the viscous dissipation term. (The $\boldsymbol{u}\cdot\boldsymbol{u}$ term on the left-hand side balances the remainder.)

3. Discussion of results

It may seem strange that the diffusion terms in (10) contribute a negligible amount to the total since diffusion plays such a critical role in the system. However, even though diffusion is essential, its main function during the evolution of the system is catalytic. In the experiments we observed that salt diffuses downward from the upper layer through the interface near the top of the lower layer thereby making the fluid just below the interface denser; that dense fluid is then redistributed by means of a clockwise circulation and ends up near the bottom. Similarly, the fluid in the upper layer that has lost salt by diffusion just above the interface becomes lighter and is redistributed by means of a clockwise circulation and ends up near the top of the upper layer. The clockwise circulations occur after diffusion has created the density differences, which are then convected to the vicinity of the boundaries much more rapidly than diffusion could move them there. That can be seen in figure 2(*b*) where the densities near the top and bottom start the same but quickly separate with lower density at the top and higher density at the bottom. (The sugar redistribution partially compensates for the salt redistribution but it is less effective because of the smaller diffusion coefficient.) The effect of diffusion across the interface continues to drive the clockwise circulations and gradually decreases the concentration differences between the two layers; overall diffusion, represented by the boundary terms in (10), also acts to decrease the differences. These effects are indicated in figures 2(*a*) and 2(*b*), which show that gradually the concentration differences from top to bottom decrease.

The function of diffusion is similar to that in Bénard convection where diffusion allows the fluid near the lower (upper) boundary to be heated (cooled) and therefore be convected upward (downward), but it is the convective process that is essentially responsible for the heat transport. So both here and in the Bénard case diffusion serves to inject buoyancy where it can be subject to convection. There is a large difference, however, since Bénard convection releases the potential energy that is made available by imposed conditions at the top and bottom boundaries of the system whereas here diffusion creates the potential energy to be released and it does so in the interior of the fluid just above and below the sloping interface. From there the clockwise circulations above and below the interface convect the unstable fluid to the vicinity of the boundaries. The part of the diffusion that is negligible is the effect from the diffusion terms themselves integrated over the entire system, apart from any convective mechanism. And the odd and counterintuitive phenomenon here is that the system ends up being even more stable than it was at the outset when it was already in a state of ostensibly minimum potential energy.†

In connection with this description it is useful to recall the concept of a thermodynamic state of a fluid. If two samples can exist in contact with each other without a change of properties, the two samples have the same state. Otherwise, their states are different. In the setup that exists at the beginning of the experiment described in this article diffusion immediately causes a change at the interface between the sugar and the salt half-layers. Therefore, the system is not in an equilibrium thermodynamic state. The energy that is released comes from the chemical potential of the fluid and is not evident in the gravitational potential energy of the initial state, although the latter is activated as soon as diffusion becomes active. The role of the chemical potential is

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† Paul Linden has suggested that the present setup is not basically different from either the finger or the diffusive one-dimensional case, where, in the absence of diffusion, nothing happens. But if one were to remove the stabilizing property, flow ensues because of gravity. In our case the same is true although without diffusion and in the absence of one of the properties the flow created is a gravity current. With both properties present in both cases diffusion generates density anomalies that release potential energy.

manifested through the diffusion terms that appear in the conservation equations of salt and sugar.

I am grateful to Marshall Ward, a fellow of the 2004 program, who came back to Woods Hole in summer 2005 to run the experiments that provided the data on the concentrations of salt and sugar that were used in this article. The article was written up at the GFD2005 program.

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